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Coherent states for the unitary symplectic group

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Abstract

We present an explicit construction of coherent states for an arbitrary irreducible representation of the unitary symplectic group $USp(4)$. Three different families of coherent states are obtained, corresponding to the subgroups $U(1) \times U(1)$, $U(2)$ and $SU(2) \times SU(2)$. The symplectic structure on the manifold of coherent states is obtained, and canonical coordinates are used to express the classical limit of quantum observables. One of the families is seen to provide a trivial classical limit.

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1. Introduction

Coherent states were first defined for the harmonic oscillator as quantum states presenting a classical-like behaviour, and later recognized to express coherence properties of the quantized radiation field [1]. These canonical coherent states are now realizable in the laboratory and largely used in quantum optics experiments. Their angular momentum analogues appeared in the early 1970s, in connection with applications to atomic physics [2], and soon a generalization to arbitrary Lie groups was presented [3].

Coherent states have proved to be a useful tool in providing not only a classical limit (i.e., real functions associated with observables and a Poisson bracket to replace the commutator) to quantum systems [4] but also a way to incorporate quantum corrections [5]. The investigation of coherent states for unitary groups $U(N)$ have attracted most of the attention [6], not only because of their long history of applications to quantum physics [7], but also because their irreducible representations are easy to manipulate [8]. Coherent states of $SU(3)$, for example, have been examined in detail [9–11].

In this paper we construct coherent states for the unitary symplectic group $USp(4)$. Symplectic groups are the natural symmetry groups of both classical and quantum mechanics, since they contain all linear transformations that preserve the canonical relations between conjugate variables. Apart from this classical theory, the symplectic groups have found applications in fields as far apart as spectroscopy [12], quantum optics [13] and the study of the genetic code [14]. Representation theory of the symplectic groups has been exaustively studied both in the compact and non-compact forms [15, 16], but matrix elements for general irreducible representations of $Sp(2N, \mathbb{C})$ for $N > 2$ are hard to manipulate. In the present case we use the Gelfand–Zetlin formalism and explicit formulas for arbitrary representations.

This work is aimed at giving $USp(4)$ a detailed treatment. We introduce coordinates in the quotient spaces associated with the subgroups $U(1) \times U(1)$, $U(2)$ and $SU(2) \times SU(2)$, which are used to parametrize the coherent states. Different coordinates are used in each case, and for symmetric representations we derive canonical coordinates, in which the symplectic 2-form on the appropriate quotient space reduces to Darboux's form.

This paper is organized as follows. In section 2 we review the properties of the symplectic group and of its subgroups. In section 3 we present the irreducible representations. Section 4 is devoted to the coherent states of general representations, while in section 5 we restrict to the symmetric ones. In section 6 we review the theory of coadjoint orbits and present the special case of $USp(4)/[SU(2) \times SU(2)]$, where the coherent states provide only the trivial representation of the group. Conclusions are presented in section 7.

2. The unitary symplectic group $USp(4)$ and its algebra

The symplectic groups are the natural symmetry groups of both classical and quantum mechanics. Classical mechanics takes place in a real manifold endowed with a closed, non-degenerate and antisymmetric two-form (a symplectic form), and the equations of motion are given by Poisson brackets ($i, j = 1, \ldots, N$)

$$\{q_i, p_j\} = \delta_{ij}. \tag{1}$$

Quantum mechanics takes place in a complex Hilbert space, and the dynamics is determined by the canonical commutation relations ($i, j = 1, \ldots, N$)

$$[\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}. \tag{2}$$

These relations can also be written in the form (now $i, j = 1, \ldots, 2N$)

$$\{\xi_i, \xi_j\} = J_{ij} \tag{3a}$$

$$[\hat{\xi}_i, \hat{\xi}_j] = \mathrm{i}\hbar J_{ij} \tag{3b}$$

where $\xi = (q_1, \ldots, q_N, p_1, \ldots, p_N)^T$ and J is the $2N \times 2N$ matrix given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4}$$

The symplectic group $Sp(2N, \mathbb{C})$ (in its defining representation) is composed of all linear complex transformations that preserve the structure of relations (3). It is easy to see therefore that

$$Sp(2N, \mathbb{C}) = \{g | gJg^T = J\}. \tag{5}$$

An equivalent definition of the symplectic group is the set of linear complex transformations that preserve the bilinear form $(\xi, \eta) = \xi^T J\eta$, where ξ, η are vectors in $\mathbb{C}^{2N}$. This complex group has two important real forms: the *normal* (non-compact) real form $Sp(2N, \mathbb{R})$, which is the one naturally associated with classical mechanics, and the *unitary* (compact) real form $USp(2N)$, which is obtained as an intersection with the unitary group

$$USp(2N) = Sp(2N, \mathbb{C}) \cap U(2N). \tag{6}$$

The Lie algebra $sp(2N, \mathbb{C})$ is the set of complex matrices X satisfying $X^T J + JX = 0$. These matrices have the general block form

$$X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \tag{7}$$

where A, B and C are complex $N \times N$ matrices, B and C being symmetric. The real symplectic group and its algebra $sp(2N, \mathbb{R})$ are obtained restricting these matrices to the real field. On the other hand, the unitary symplectic group and its algebra $usp(2N)$ are obtained by imposing X to be anti-Hermitian, $X^\dagger = -X$, or equivalently

$$A^\dagger = -A \qquad B^\dagger + C = 0 = C^\dagger + B. \tag{8}$$

The ten-dimensional symplectic algebra $usp(4)$ has, in the Cartan–Weyl scheme, the following commutation relations:

$$[H_i, H_j] = 0 \qquad i, j = 1, 2 \tag{9}$$

$$\left[H_i, E_j^\pm\right] = \pm(\alpha_j)_i E_j^\pm \qquad i = 1, 2 \qquad j = 1, \ldots, 4 \tag{10}$$

$$\left[E_i^+, E_i^-\right] = (\alpha_i)_1 H_1 + (\alpha_i)_2 H_2 \tag{11}$$

$$\left[E_1^+, E_2^+\right] = \sqrt{2} E_3^+ \qquad \left[E_1^+, E_3^+\right] = \sqrt{2} E_4^+ \tag{12}$$

where H_1 and H_2 span the abelian Cartan subalgebra and $E_i^- = \left(E_i^+\right)^\dagger$ (the Killing form is negative definite on $usp(2N)$, as usual). The positive roots are

$$\alpha_1 = [1, -1] \qquad \alpha_2 = [0, 2] \qquad \alpha_3 = [1, 1] \qquad \alpha_4 = [2, 0]. \tag{13}$$

Note the existence of two $su(2)$ algebras generated by $\left\{H_1, E_4^+, E_4^-\right\}$ and $\left\{H_2, E_2^+, E_2^-\right\}$. The direct sum of these algebras is a maximal subalgebra of $usp(4)$ that is associated with the canonical symmetry breaking chain

$$usp(4) \supset su(2) \oplus su(2) \supset u(1) \oplus u(1). \tag{14}$$

This chain can be used to provide a complete set of quantum numbers that label uniquely the vectors in any ireducible representation of $usp(4)$ [17]. Matrix elements for the algebra elements in an arbitrary irrep were obtained in [18] using these quantum numbers and implemented for algebraic computation through the package Killing [19]. We present the main results in the next section.

A general element g of $Sp(4, \mathbb{C})$ is obtained as an exponential of the kind

$$g = \exp\left\{\gamma_1 H_1 + \gamma_2 H_2 + \sum_{i=1}^{4} \eta_i E_i^+ + \rho_i E_i^-\right\} \tag{15}$$

where all Greek letters represent arbitrary complex numbers. As already noted, this noncompact complex group has two real forms of (real) dimension 10: a noncompact one, denoted $Sp(4, \mathbb{R})$, obtained by restricting all the previous parameters to the real field; a compact one, denoted $USp(4)$, which consists of unitary matrices obtained by imposing $\rho_i = -\eta_i^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. In this paper we will always consider this compact case.

In what follows we will use the Gauss decomposition of $USp(4)$ and its quotient spaces. To perform that decomposition one must find functions $(\tau, \tilde{\tau}, \alpha)$, depending only on the group coordinates, such that the group elements can be decomposed as

$$g = \exp\left\{\sum_{i=1}^{4} \tau_i E_i^+\right\} \exp\{\alpha_1 H_1 + \alpha_2 H_2\} \exp\left\{\sum_{i=1}^{4} \tilde{\tau}_i E_i^-\right\}. \tag{16}$$

Moreover, the exponential involving only positive (or negative) roots can also be disentangled. It is known that if A, B are elements of a Lie algebra, then

$$\mathrm{e}^{A+B} = \mathrm{e}^{A}\,\mathrm{e}^{B}\,\mathrm{e}^{C_1}\,\mathrm{e}^{C_2}\ldots \tag{17}$$

where all C_i belong to the Lie algebra. Relations of this kind are usually called Baker–Campbell–Hausdorff formulas. A recurrence procedure to find the C_i functions in terms of commutators can be found in [20]. The sequence of exponentials is finite if the algebra is nilpotent, which is the case for the positive (negative) roots. We shall need only the first two elements:

$$C_1 = -\tfrac{1}{2}[A, B] \qquad C_2 = -\tfrac{1}{3}[[A, B], B] - \tfrac{1}{6}[[A, B], A]. \tag{18}$$

Therefore, a general group element can be written in terms of new coordinates $(z, \tilde{z}, \beta)$ as

$$g = \left\{\prod_{i=1}^{4} \exp z_i E_i^+\right\} \exp\{\beta_1 H_1 + \beta_2 H_2\} \left\{\prod_{i=1}^{4} \exp \tilde{z}_i E_i^-\right\}. \tag{19}$$

3. Irreducible representations of $usp(4)$

Irreducible representations of $usp(4)$ are labelled by two integers, λ_1 and λ_2, with $\lambda_1 > \lambda_2$. The basis vectors analogous to the Gelfand–Tsetlin basis for $U(N)$ are given by $|\sigma_1, \sigma_2, h_1, h_2\rangle$, where σ_i are positive integers obeying the inequalities [17]

$$\lambda_1 - \lambda_2 \leqslant \sigma_1 + \sigma_2 \leqslant \lambda_1 + \lambda_2 \tag{20}$$

$$\lambda_2 - \lambda_1 \leqslant \sigma_1 - \sigma_2 \leqslant \lambda_1 - \lambda_2 \tag{21}$$

and $h_i = -\sigma_i, -\sigma_i + 2, \ldots, \sigma_i - 2, \sigma_i$. The quantum numbers (σ_1, σ_2) are the highest weights of the $su(2) \oplus su(2)$ irreducible representations.

By construction, the diagonal operators have simple matrix elements,

$$H_i|\sigma_1, \sigma_2, h_1, h_2\rangle = h_i|\sigma_1, \sigma_2, h_1, h_2\rangle \tag{22}$$

$$J_i|\sigma_1, \sigma_2, h_1, h_2\rangle = \sigma_i(\sigma_i + 2)|\sigma_1, \sigma_2, h_1, h_2\rangle \tag{23}$$

where the Casimir operators of the $sp(2)$ subalgebras are

$$J_1 = H_1^2 + \left[E_4^+, E_4^-\right]_+ \qquad J_2 = H_2^2 + \left[E_2^+, E_2^-\right]_+ \tag{24}$$

and $[A, B]_+ = AB + BA$.

The action of the ladder operators $\left\{E_2^{\pm}, E_4^{\pm}\right\}$ are similar to those of the angular momentum algebra and are given by

$$E_2^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = |\sigma_1, \sigma_2, h_1, h_2 \pm 2\rangle \tag{25}$$

$$E_4^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = |\sigma_1, \sigma_2, h_1 \pm 2, h_2\rangle \tag{26}$$

Note that these operators do not couple different representations of the $sp(2)$ subalgebras. That is not the case with the remaining elements, $\left\{E_1^{\pm}, E_3^{\pm}\right\}$, which may couple the first neighbourhood $(\sigma_i \pm 1)$ representations. Their actions are given by

$$\begin{aligned} E_1^{\pm}|\sigma_1, \sigma_2, h_1, h_2\rangle = {} & A^{\pm}|\sigma_1 + 1, \sigma_2 + 1, h_1 \pm 1, h_2 \mp 1\rangle \\ & \pm B^{\pm}|\sigma_1 + 1, \sigma_2 - 1, h_1 \pm 1, h_2 \mp 1\rangle \\ & \mp C^{\pm}|\sigma_1 - 1, \sigma_2 + 1, h_1 \pm 1, h_2 \mp 1\rangle \\ & + D^{\pm}|\sigma_1 - 1, \sigma_2 - 1, h_1 \pm 1, h_2 \mp 1\rangle \end{aligned} \tag{27}$$

and

$$\begin{aligned} E_3^{\pm}|\sigma_1,\sigma_2,h_1,h_2\rangle = {} & A'^{\pm}|\sigma_1+1,\sigma_2+1,h_1\pm 1,h_2\pm 1\rangle \\ & \pm B'^{\pm}|\sigma_1+1,\sigma_2-1,h_1\pm 1,h_2\pm 1\rangle \\ & \mp C'^{\pm}|\sigma_1-1,\sigma_2+1,h_1\pm 1,h_2\pm 1\rangle \\ & + D'^{\pm}|\sigma_1-1,\sigma_2-1,h_1\pm 1,h_2\pm 1\rangle \end{aligned} \tag{28}$$

where the primed coefficients are obtained from the unprimed ones just by exchanging the sign of h_2,

$$X'^{\pm} = X^{\pm}(h_2 \to -h_2) \qquad X \in \{A, B, C, D\}. \tag{29}$$

The coefficients A^+ and B^+ can be shown to be [18]

$$A^+ = \left\{ \frac{(\lambda_+ - \sigma_+)(\lambda_+ + \sigma_+ + 6)(\sigma_+ - \lambda_- + 2)(\sigma_+ + \lambda_- + 4)(\sigma_1 + h_1 + 2)(\sigma_2 - h_2 + 2)}{64(\sigma_1+1)(\sigma_1+2)(\sigma_2+1)(\sigma_2+2)} \right\}^{\frac{1}{2}} \tag{30}$$

$$B^+ = \left\{ \frac{(\lambda_- + \sigma_- + 2)(\lambda_- - \sigma_-)(\lambda_+ - \sigma_- + 2)(\lambda_+ + \sigma_- + 4)(\sigma_1 + h_1 + 2)(\sigma_2 + h_2)}{64(\sigma_1+1)(\sigma_1+2)\sigma_2(\sigma_2+1)} \right\}^{\frac{1}{2}} \tag{31}$$

where $\lambda_\pm = \lambda_1 \pm \lambda_2$ and $\sigma_\pm = \sigma_1 \pm \sigma_2$. The C^+ and D^+ coefficients are related to the previous ones by simple changes in the arguments,

$$C^+(\sigma_1,\sigma_2,h_1,h_2) = B^+(\sigma_1-1,\sigma_2-1,-(h_1+1),-(h_2-1)) \tag{32}$$

$$D^+(\sigma_1,\sigma_2,h_1,h_2) = A^+(\sigma_1-1,\sigma_2+1,-(h_1+1),-(h_2-1)). \tag{33}$$

Moreover, the lowering coefficients can be obtained from the raising ones as follows:

$$A^-(\sigma_1,\sigma_2,h_1,h_2) = D^+(\sigma_1+1,\sigma_2+1,h_1-1,h_2+1) \tag{34}$$

$$B^-(\sigma_1,\sigma_2,h_1,h_2) = C^+(\sigma_1+1,\sigma_2-1,h_1-1,h_2+1) \tag{35}$$

$$C^-(\sigma_1,\sigma_2,h_1,h_2) = B^+(\sigma_1-1,\sigma_2+1,h_1-1,h_2+1) \tag{36}$$

$$D^-(\sigma_1,\sigma_2,h_1,h_2) = A^+(\sigma_1-1,\sigma_2-1,h_1-1,h_2+1). \tag{37}$$

4. Coherent states

4.1. Definition

Given an irreducible representation $[\lambda_1, \lambda_2]$, we define coherent states $|z\rangle$ with respect to the lowest weight $|0\rangle = |\lambda_1, \lambda_2, -\lambda_1, -\lambda_2\rangle$ by

$$|z\rangle = \exp\left\{z_1 E_1^+\right\} \exp\left\{z_2 E_2^+\right\} \exp\left\{z_3 E_3^+\right\} \exp\left\{\tfrac{1}{\sqrt{2}}(z_4 - z_1 z_3) E_4^+\right\}|0\rangle \tag{38}$$

The four complex numbers z_i are coordinates for the manifold of coherent states, and we made an appropriate choice for z_4 for later convenience.

It is easy to see that in the symmetric representations, where $\lambda_1 = \lambda$ and $\lambda_2 = 0$, any element of the kind $\exp\left\{z_2 E_2^+\right\}$ leaves the lowest weight invariant

$$\exp\left\{z_2 E_2^+\right\}|\lambda, 0, -\lambda, 0\rangle = |\lambda, 0, -\lambda, 0\rangle. \tag{39}$$

This vector has therefore $U(2)$ as its isotropy group, while in the non-symmetric case the isotropy of the lowest weight is given just by the $U(1) \times U(1)$ group generated by the Cartan subalgebra (we come back to this subject in section 5). To avoid this isotropy we impose the restriction $z_2 = 0$ on (38) when dealing with symmetric representations, analysed in full detail in section 4.

Coherent states related to the quotient space $USp(4)/[SU(2) \times SU(2)]$ can only be contructed in antisymmetric representations, using a non-minimal weight as reference vector. In that case the decomposition formula (19) is no longer useful, and we have to return to equation (15). Section 6.1 is devoted to this discussion.

4.2. General properties, resolution of unity

Coherent states provide an overcomplete family of states inside a given $[\lambda_1, \lambda_2]$ irreducible representation, of dimension $D_{\lambda_1,\lambda_2} = (\lambda_1+2)(\lambda_2+1)(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+3)/6$. A resolution of unity

$$\int \mathrm{d}\mu(z)|z\rangle\langle z| = \sum |\sigma_1, \sigma_2, h_1, h_2\rangle\langle\sigma_1, \sigma_2, h_1, h_2| = \mathbf{1} \tag{40}$$

(where the sum includes all basis vectors) exists with $\mathrm{d}\mu(z)$ an appropriate measure obtained from the Haar measure of the group expressed in the z coordinates of (38). The determination of the measure is in general a difficult problem, but we shall present it explicitly for symmetric representations in the next section.

On the other hand, let the coherent states be written in terms of the basis states of the representation as

$$|z\rangle = \sum \mathcal{F}_n(z)|\sigma_1, \sigma_2, h_1, h_2\rangle \tag{41}$$

where the sum again includes all basis states and n represents the quantum numbers. The functions $\mathcal{F}_n(z)$ are a family of polynomials spanning a Hilbert space of dimension D_{λ_1,λ_2}, analogous to the usual Bargmann representation. The scalar product in this space is derived from the resolution of unity (40):

$$\int \mathrm{d}\mu(z)\mathcal{F}_n^*(z)\mathcal{F}_{n'}(z) = \delta_{nn'}. \tag{42}$$

Another interesting property of this Hilbert space is the existence of a reproducing kernel $K(z, z')$ such that

$$\int \mathrm{d}\mu(z)\mathcal{F}_n(z)K(z, z') = \mathcal{F}_n(z'). \tag{43}$$

This kernel is just the overlap of two coherent states

$$K(z, z') = \langle z|z'\rangle \tag{44}$$

and its determination from the definition (38) is also in general a nontrivial task. We present it only for the symmetric representations.

4.3. Action of algebra elements on coherent states

The action of a $usp(4)$ element on a coherent state does not, in general, yield another coherent state. From the commutation relations and the definition (38), we can find a differential representation for the algebra elements. It is not difficult to see that the raising operators and the diagonal ones are represented by

$$E_1^+|z\rangle = [\partial_1 + z_3\partial_4]|z\rangle \tag{45}$$

$$E_2^+|z\rangle = [\partial_2 - \sqrt{2}z_1\partial_3]|z\rangle \tag{46}$$

$$E_3^+|z\rangle = [\partial_3 - z_1\partial_4]|z\rangle \tag{47}$$

$$E_4^+|z\rangle = \sqrt{2}\partial_4|z\rangle \tag{48}$$

$$H_1|z\rangle = [z_1\partial_1 + z_3\partial_3 + 2z_4\partial_4 - \lambda_1]|z\rangle \tag{49}$$

$$H_2|z\rangle = [-z_1\partial_1 + 2z_2\partial_2 + z_3\partial_3 - \lambda_2]|z\rangle \tag{50}$$

where ∂_i indicates a derivative with respect to z_i. The lowering operators are a bit more complicated,

$$E_1^-|z\rangle = \left[-z_1^2\partial_1 + \sqrt{2}(z_3 + \sqrt{2}z_1z_2)\partial_2 + (z_4 - z_1z_3)\partial_3 - z_1z_4\partial_4 + z_1(\lambda_1 - \lambda_2)\right]|z\rangle \tag{51}$$

$$E_2^-|z\rangle = \left[-\sqrt{2}z_3\partial_1 - 2z_2^2\partial_2 + 2z_2\lambda_2\right]|z\rangle \tag{52}$$

$$\begin{aligned} E_3^-|z\rangle = \big[&-(z_4 + z_1z_3)\partial_1 - 2z_2(z_3 + \sqrt{2}z_1z_2)\partial_2 - z_3^2\partial_3 \\ &- z_3z_4\partial_4 + (z_3 + 2\sqrt{2}z_1z_2)\lambda_2 + z_3\lambda_1\big]|z\rangle \end{aligned} \tag{53}$$

$$\frac{E_4^-}{\sqrt{2}}|z\rangle = \left[-z_1z_4\partial_1 - \frac{1}{\sqrt{2}}(z_3 + \sqrt{2}z_1z_2)^2\partial_2 - z_3z_4\partial_3 - z_4^2\partial_4 + z_4\lambda_1 + z_1(z_3 + z_1z_2)\lambda_2\right]|z\rangle. \tag{54}$$

Since in the symmetric representation the coherent states do not depend on z_2, the derivatives with respect to this variable vanish in that case. The hermitian adjoint properties of these operators must be analysed in relation to the scalar product (42).

4.4. Normalization

The states defined above are not normalized. In order to obtain their norm, we need to determine the variables α and F in the following equation:

$$\langle z|z\rangle = \langle 0|D(z_1, z_2, z_3, (z_4 - z_1z_3)/\sqrt{2})D^\dagger(z_1, z_2, z_3, (z_4 - z_1z_3)/\sqrt{2})|0\rangle \tag{55}$$

$$= \langle 0|D^\dagger(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\exp\{F_1H_1 + F_2H_2\}D(\alpha_1, \alpha_2, \alpha_3, \alpha_4)|0\rangle \tag{56}$$

where

$$D^\dagger(z_1, z_2, z_3, z_4) = \exp\left\{z_1E_1^+\right\}\exp\left\{z_2E_2^+\right\}\exp\left\{z_3E_3^+\right\}\exp\left\{z_4E_4^+\right\}. \tag{57}$$

Note that the order of the raising and lowering operators was reversed. The action upon the lowest weight then becomes trivial and the diagonal term provides the required normalization. The calculation can be done explicitly in the fundamental representation. The final result is

$$\alpha_1 = f^{-1}(z_1 - z_3^*z_4) - \sqrt{2}z_2^*\alpha_3 \tag{58}$$

$$\alpha_2 = \tfrac{1}{\sqrt{2}}(g - f|\alpha_1|^2)^{-1}[\sqrt{2}z_2 + z_1^*(z_3 + \sqrt{2}z_1z_2) + f\alpha_1\alpha_3^*] \tag{59}$$

$$\alpha_3 = f^{-1}(z_3 + z_1^*z_4) \tag{60}$$

$$\alpha_4 = \sqrt{2} f^{-1} z_4 - \sqrt{2}\alpha_1\alpha_3 \tag{61}$$

$$F_1 = -\ln f \qquad F_2 = -\ln(g - f|\alpha_1|^2) \tag{62}$$

where f and g are real numbers given by

$$f = 1 + |z_1|^2 + |z_3|^2 + |z_4|^2 \tag{63}$$

$$g = 1 + 2|z_2|^2 + |z_3 + \sqrt{2} z_1 z_2|^2 \tag{64}$$

The norm is then found to be

$$\langle z|z\rangle = f^{\lambda_1}(g - f|\alpha_1|^2)^{\lambda_2}. \tag{65}$$

It is always possible to define, on the manifold of coherent states, a symplectic 2-form [21]

$$\omega = \mathrm{i}\hbar \sum_{i,j=1}^{4} \omega_{ij}\, \mathrm{d}z_i \wedge \mathrm{d}z_j^* \tag{66}$$

and the associated Poisson bracket

$$\{f, g\} = \frac{1}{\mathrm{i}\hbar} \sum_{i,j} \omega^{ij} \left\{ \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j^*} - \frac{\partial f}{\partial z_j^*} \frac{\partial g}{\partial z_i} \right\} \tag{67}$$

where $\omega^{ik}\omega_{kj} = \delta^i_j$ and the elements ω_{ij} are given in terms of the norm [9, 22]

$$\omega_{ij} = \frac{\partial^2 \ln\langle z|z\rangle}{\partial z_i \partial z_j^*}. \tag{68}$$

Given X and Y two elements of $usp(4)$, the Poisson bracket of their mean values is related to their commutator by

$$\hbar\{\langle z\|X\|z\rangle, \langle z\|Y\|z\rangle\} = \mathrm{i}\langle z\|[X, Y]\|z\rangle \tag{69}$$

(where $\|z\rangle$ denotes the normalized coherent states) and can be used to obtain a well-defined classical limit for a quantum system whose observables can be written in terms of $usp(4)$ elements.

4.5. Classical Limit

We now want to obtain the classical limit of a Hamiltonian defined on a irreducible representation of $Sp(4)$. This limit corresponds to letting $\hbar \to 0$ and simultaneously increasing the size of the representation so that the density of states may increase. Since the representation is labelled by two integer numbers $[\lambda_1, \lambda_2]$, the actual classical limit can be attained in many different ways [9, 23]. One can for example consider only symmetric representations, by letting λ_1 go to infinity while keeping $\lambda_2 = 0$. Or one can let both λ_1 and λ_2 go to infinity and keep their ratio λ_1/λ_2 constant. In any case the limit can be implemented by imposing the Planck constant $\hbar$ to be proportional to $(\lambda_1 + \lambda_2)^{-1}$.

If we take the expectation value with respect to normalized coherent states $\|z\rangle$ of the Heisenberg equation of motion

$$\langle z\|\frac{\mathrm{d}X}{\mathrm{d}t}\|z\rangle = \frac{\mathrm{i}}{\hbar}\langle z\|[H, X]\|z\rangle \tag{70}$$

where H is the Hamiltonian and X is an $usp(4)$ element, and multiply both sides by $\hbar$, its classical limit is well-defined and easily seen to be

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}t} = \{\mathcal{H}, \mathcal{X}\} \tag{71}$$

where $\mathcal{H}$ and $\mathcal{X}$ are the classical limits of the corresponding operators, defined as

$$\mathcal{H} = \lim_{\hbar \to 0} \langle z \| H \| z \rangle. \tag{72}$$

Moreover, expectation values of products of operators factorize [5, 9]

$$\langle z \| \hbar X_1 \hbar X_2 \| z \rangle \to \langle z \| \hbar X_1 \| z \rangle \langle z \| \hbar X_2 \| z \rangle \qquad \text{for} \quad \hbar \to 0 \tag{73}$$

and therefore the connection between commutators and Poisson brackets is not restricted to observables linear in the algebra elements. In the following section the classical limit is treated more explicitly.

5. Symmetric representations

As already mentioned, in the symmetric representations, for which $\lambda_1 = \lambda$ and $\lambda_2 = 0$, we have to impose the restriction $z_2 = 0$. The norm of the coherent states greatly simplifies in this case and reduces to

$$\langle z | z \rangle = (1 + |z_1|^2 + |z_3|^2 + |z_4|^2)^\lambda. \tag{74}$$

This simplicity will allow us to explore many properties of the coherent states explicitly. Using the Bargmann-like representation we derive the appropriate measure and the related resolution of unity. We also introduce canonical coordinates that bring the Poisson bracket and the symplectic 2-form to real diagonal form.

5.1. Resolution of unity

In the symmetric representations H_1, H_2 and J_2 are enough to identify all vectors. We can use their differential representation to obtain the following partial differential equations for $\mathcal{F}_n(z)$ (note that in the symmetric representations the variable z_2 vanishes identically and that $\sigma_2 = \lambda - \sigma_1$):

$$[z_1\partial_1 + z_3\partial_3 + 2z_4\partial_4 - \lambda]\mathcal{F}_n(z) = h_1\mathcal{F}_n(z) \tag{75}$$

$$[-z_1\partial_1 + z_3\partial_3]\mathcal{F}_n(z) = h_2\mathcal{F}_n(z) \tag{76}$$

$$[(z_1\partial_1)^2 + (z_3\partial_3)^2 + 2(z_3\partial_1)(z_1\partial_3) + 2z_3\partial_3]\mathcal{F}_n(z) = \sigma_2(\sigma_2 + 2)\mathcal{F}_n(z). \tag{77}$$

The solution is simple:

$$\mathcal{F}_n(z) = c z_1^{(\sigma_2 - h_2)/2} z_3^{(\sigma_2 + h_2)/2} z_4^{(\sigma_1 + h_1)/2} \tag{78}$$

where c is a constant, whose value is fixed by normalization. One can show that it involves only binomials:

$$c^2 = \binom{\lambda}{\sigma_1} \binom{\sigma_2}{(\sigma_2 + h_2)/2} \binom{\sigma_1}{(\sigma_1 + h_1)/2}. \tag{79}$$

From this explicit form of the functions $\mathcal{F}_n(z)$ we are able to determine the overlap of two coherent states as

$$\langle z | z' \rangle = (1 + z_1^* z_1' + z_3^* z_3' + z_4^* z_4')^\lambda \tag{80}$$

and to infer the measure:

$$\mathrm{d}\mu(z) = \frac{(\lambda+1)(\lambda+2)(\lambda+3)}{\pi^3} \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_3\, \mathrm{d}^2 z_4}{(1+|z_1|^2+|z_3|^2+|z_4|^2)^5}. \tag{81}$$

Note that $(\lambda+1)(\lambda+2)(\lambda+3)$ is the dimension of the representation. This is the Haar measure on $USp(4)/U(2)$ and is very similar to previous results on Lie groups [9, 25]. Both the orthogonality relations (42) and the reproducing kernel property (43) can therefore be verifyed explicitly in the symmetric representations.

5.2. Poisson bracket and canonical coordinates

In the case of symmetric representations we obtain ($\lambda \equiv \lambda_1$)

$$\omega_{ij} = \frac{\lambda}{f}\left(\delta_{ij} - \frac{z_j z_i^*}{f}\right) \tag{82}$$

(δ_{ij} is the delta of Kronecker) with $i, j = 1, 3, 4$, and the inverse matrix is given by

$$\omega^{ij} = \frac{f}{\lambda}(\delta_{ij} + z_j z_i^*). \tag{83}$$

This matrix can be used to define the Poisson bracket

$$\{f, g\} = \frac{1}{\mathrm{i}\hbar} \sum_{i,j=1,3,4} \omega^{ij} \left\{ \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_j^*} - \frac{\partial f}{\partial z_j^*}\frac{\partial g}{\partial z_i} \right\}. \tag{84}$$

We can now bring this expression to a diagonal form, customary in classical mechanics,

$$\{f, g\} = \sum_{i=1}^{3} \left\{ \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right\} \tag{85}$$

by defining the canonical coordinates

$$\frac{q_1 + \mathrm{i}p_1}{\sqrt{2\Xi}} = \frac{z_1}{\sqrt{1+|z_1|^2+|z_3|^2+|z_4|^2}} \tag{86}$$

and

$$\frac{q_j + \mathrm{i}p_j}{\sqrt{2\Xi}} = \frac{z_{j+1}}{\sqrt{1+|z_1|^2+|z_3|^2+|z_4|^2}} \qquad j = 2, 3 \tag{87}$$

where we have used the variable $\Xi := \hbar\lambda$, which is adequate for treating the classical limit.

The symplectic 2-form in these new coordinates is written simply

$$\omega = \sum_{i=1}^{3} \mathrm{d}q_i \wedge \mathrm{d}p_i. \tag{88}$$

The results expressed in equations (85) and (88) allow us to call the (q_i, p_i) pairs canonical coordinates. Note that they have the constraint

$$E(q, p) = \sum_{i=1}^{3} q_i^2 + p_i^2 \leqslant 2\Xi \tag{89}$$

which expresses the compactness of the group $USp(4)$ and of its associated quotient spaces.

In these coordinates, the expectation values of the algebra elements with respect to normalized coherent states can be obtained:

$$\mathcal{E}_1^+ = \langle z\|\hbar E_1^+\|z\rangle = \frac{F}{2}(q_1 - \mathrm{i}p_1) + \frac{1}{2}(q_2 + \mathrm{i}p_2)(q_3 - \mathrm{i}p_3) \tag{90}$$

$$\mathcal{E}_2^+ = \langle z\|\hbar E_2^+\|z\rangle = \frac{1}{\sqrt{2}}(-q_2 + \mathrm{i}p_2)(q_1 + \mathrm{i}p_1) \tag{91}$$

$$\mathcal{E}_3^+ = \langle z\|\hbar E_3^+\|z\rangle = \frac{F}{2}(q_2 - \mathrm{i}p_2) - \frac{1}{2}(q_1 + \mathrm{i}p_1)(q_3 - \mathrm{i}p_3) \tag{92}$$

$$\mathcal{E}_4^+ = \langle z\|\hbar E_4^+\|z\rangle = \frac{F}{\sqrt{2}}(q_3 - \mathrm{i}p_3) \tag{93}$$

$$\mathcal{H}_1 = \langle z\|\hbar H_1\|z\rangle = \frac{1}{2}\left(q_3^2 + p_3^2 - F^2\right) \tag{94}$$

$$\mathcal{H}_2 = \langle z\|\hbar H_2\|z\rangle = \frac{1}{2}\left(q_2^2 + p_2^2 - q_1^2 - p_1^2\right) \tag{95}$$

where $F = \sqrt{2\Xi - E}$. The expected values for the negative roots are obtained simply by complex conjugation. It is easy to see that the classical functions have Poisson brackets (as given by (85)) compatible with the *usp*(4) commutation relations and (69).

The classical limit is now implemented by letting $\hbar \to 0, \lambda \to \infty$, while keeping the product $\hbar\lambda$ constant. As classical observables we take the above expectation values of the algebra elements in the coherent states. Note that the Planck constant is present on the left-hand side (the 'quantum' side) of (90)–(96) in order to provide a dimensional meaning, while the right-hand side (the 'classical' side) is independent of it.

6. Coherent states and coadjoint orbits

The relation between coherent states and coadjoint orbits of Lie groups has been long known, and the literature on the subject (and its connection with the theory of geometric quantization) is vast. We provide here only a brief description of the theory and outline some of the main results, not going into many details. Recent accounts and many references can be found in [9, 26–28].

Let G be a Lie group with a (left) action on a smooth manifold X. Let this action be denoted gx (where $g \in G$ and $x \in X$) and assumed to be continuous in both variables. If this G-action is transitive, which means that given $x, y \in X$ there is always $g \in G$ such that $x = gy$, then X is said to be a homogeneous space. All such spaces are diffeomorphic to quotient spaces G/H, where H is a closed subgroup of G (elements of G/H are of the form gH, and G has a natural action on this space given by $gH \mapsto g'gH$).

In order to see this, take a fixed point $x_0 \in X$ and let H_0 be its stability group, given by $H_0 = \{h \in G | hx_0 = x_0\}$. Since X is homogeneous, for any $x \in X$ there is $g \in G$ for which $x = gx_0$. The identification $x \mapsto gH_0$ is then a diffeomorphism between X and G/H_0. In fact, given a general space X, the orbit of $x \in X$ under G is the set $Gx = \{y \in X | g \in G\}$ and thus X consists of a collection of disjoint G-orbits, each of them being a homogeneous space.

On the other hand, G has a natural action, the adjoint action, on its algebra $\mathfrak{g}$, in which $A \mapsto g(A) = gAg^{-1}$ for any $A \in \mathfrak{g}$ (since we are only concerned with matrix groups, the product gA can be regarded simply as a matrix multiplication). This induces an action of G on $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, in which a point $\rho \in \mathfrak{g}^*$ is mapped, under the action of a group element g, to the point ρ_g. This is called the coadjoint action, and ρ_g is defined by $\rho_g(A) = \rho(g(A))$, $A \in \mathfrak{g}$.

The set $\mathcal{O}_\rho = \{\rho_g | g \in G\}$ is called the coadjoint orbit of ρ under G. This is a homogeneous space and therefore diffeomorphic to the quotient G/H_ρ, where $H_\rho = \{g \in G | \rho_g = \rho\}$ is the stability group of ρ. It is clear that, given a fixed ρ, its coadjoint orbit can be used to define a representation $\mathcal{R}_\rho$ of the algebra $\mathfrak{g}$ as functions on G/H_ρ, in which

$$\mathcal{R}_\rho(A)(x) = \rho_x(A) \qquad A \in \mathfrak{g} \qquad x \in G/H_\rho. \tag{96}$$

This representation is naturally endowed with a Poisson bracket $\{\cdot,\cdot\}$ induced by the algebra Lie product $[\cdot,\cdot]$ according to $\{\mathcal{R}_\rho(A), \mathcal{R}_\rho(B)\}(x) = \rho_x([A, B])$. In local coordinates such Poisson bracket is given as in (67).

We have thus seen that quotient spaces G/H can be regarded as phase spaces, and that the Lie algebra $\mathfrak{g}$ corresponding to G can be represented by functions on this space. Let us now make the connection with coherent states. Let $\mathbb{H}$ be a Hilbert space that carries a unitary irreducible representation of $\mathfrak{g}$, denoted T. To any vector $|0\rangle \in \mathbb{H}$ there corresponds an element $\rho \in \mathfrak{g}^*$ whose action on $\mathfrak{g}$ is given by $\rho(A) = \langle 0|T(A)|0\rangle$, $A \in \mathfrak{g}$. Therefore, there is a natural identification between the coadjoint orbit ρ_g and the set of vectors $|g\rangle = T(g)|0\rangle$, in which $\rho_g(A) = \langle g|T(A)|g\rangle = \langle 0|T(g(A))|0\rangle$.

The set of coherent states $|g\rangle$, $g \in G$, is therefore topologically equivalent to the quotient G/H, where H is the isotropy group of the fiducial vector $|0\rangle$. This vector is usually taken to be of highest (or lowest) weight (coherent states built up from a general weight share most of the properties of the usual ones [1, 3]). In the case of *usp*(4), we have to distinguish between non-symmetric and symmetric representations. In the first case, the isotropy group of the lowest state is isomorphic to $U(1) \times U(1)$ and therefore the coherent states are labelled by points in $USp(4)/[U(1) \times U(1)]$. In the symmetric representations the coherent states were again taken to be the orbit of the lowest weight, which in this case was diffeomorphic to $USp(4)/U(2)$.

The quotient of $USp(4)$ by its maximal subgroup $SU(2) \times SU(2)$ was not found in this context. The reason for this absence is the fact that $SU(2) \times SU(2)$ does not arise as isotropy subgroup of lowest weights. The only vectors that have this group as an isotropy are of the kind $|0, 0, 0, 0\rangle$, which can only be found in antisymmetric representations ($\lambda_1 = \lambda_2$), because of inequalities (20). This case is analysed in the next section.

6.1. The maximal subgroup $SU(2) \times SU(2)$

The algebra of the first subgroup $SU(2)$ consists in $\{H_1, E_4^+, E_4^-\}$. The other subgroup $SU(2)$ is generated by $\{H_2, E_2^+, E_2^-\}$. Therefore, the remaining elements of the *usp*(4) algebra generate the coset:

$$USp(4)/[SU(2) \times SU(2)] = M(\eta) = \exp\left\{\eta_1 E_1^+ + \eta_2 E_3^+ - \eta_1^* E_1^- - \eta_2^* E_3^-\right\}. \tag{97}$$

In the fundamental 4×4 representation we have

$$\eta_1 E_1^+ + \eta_2 E_3^+ - \eta_1^* E_1^- - \eta_2^* E_3^- = \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} \tag{98}$$

where B is a 2×2 matrix given by

$$B = \begin{pmatrix} -\eta_1 & -\eta_2^* \\ \eta_2 & -\eta_1^* \end{pmatrix}. \tag{99}$$

The exponential can easily be done, and we obtain

$$M(\eta) = \begin{pmatrix} \mathbf{1}\cos|\eta| & W \\ -W^\dagger & \mathbf{1}\cos|\eta| \end{pmatrix} \tag{100}$$

where $\mathbf{1}$ is the 2×2 identity matrix, $|\eta|^2 = |\eta_1|^2 + |\eta_1|^2$ and W is given by

$$W = \frac{\sin|\eta|}{|\eta|} B. \tag{101}$$

Note that $\mathbf{1}\cos|\eta| = \sqrt{\mathbf{1} - W^\dagger W} = \sqrt{\mathbf{1} - WW^\dagger}$. Variables W are called projective coordinates of the coset representative, and the action of the group upon this space becomes a holomorphic transformation [25].

Coherent states $|\eta\rangle$ are the orbit of a fiducial state $|0\rangle$ under the action of the quotient space G/H, $|\eta\rangle = M(\eta)\,|0\rangle$. In order to construct coherent states that are parametrized by points in $USp(4)/[SU(2)\times SU(2)]$, we need to find a fiducial state $|0\rangle$ that has $SU(2)\times SU(2)$ as its isotropy group. States with that property appear only in antisymmetric representations, and in Gelfand–Tsetlin notation they are given by $|0,0,0,0\rangle$. This state, which is not a maximum weight, is annihilated not only by all long roots $\left\{E_2^{\pm}, E_4^{\pm}\right\}$, but also by the diagonal operators H_1 and H_2,

$$H_1|0\rangle = H_2|0\rangle = 0. \tag{102}$$

This has an important consequence. The average value of any algebra element A in the coherent state $|\eta\rangle$ (not the same as the previous $|z\rangle$) is

$$\langle\eta|A|\eta\rangle = \langle 0|M^{-1}(\eta)AM(\eta)|0\rangle \tag{103}$$

where we have used the fact that the representations are unitary. Since $M^{-1}AM$ is also an element of the $usp(4)$ algebra, it can be written as a general linear combination of all roots. But the average value of a non-zero root in a basis state is always zero. Therefore, it is clear from (102) that the average value of any element of $usp(4)$ will vanish in the coherent state:

$$\langle\eta|A|\eta\rangle = 0 \qquad \forall A\in usp(4) \qquad \forall\eta\in\mathbb{C}. \tag{104}$$

We see that the coherent states provide only a trivial representation of the group in this case, in which every element is represented by the identity operation.

Let us examine the details of a particular exemple. The simplest case of antisymmetric representation is $\lambda_1 = \lambda_2 = 1$, a representation of dimension 5. In that case we have

$$\eta_1 E_1^+ + \eta_2 E_3^+ - \eta_1^* E_1^- - \eta_2^* E_3^- = \begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} \tag{105}$$

where B is now given by a line,

$$B = \sqrt{2}(\eta_2 \quad \eta_1 \quad -\eta_1^* \quad \eta_2^*). \tag{106}$$

The exponential can be shown to be

$$M(\eta) = \exp\begin{pmatrix} 0 & B \\ -B^\dagger & 0 \end{pmatrix} = \begin{pmatrix} \cos 2|\eta| & W \\ -W^\dagger & \sqrt{\mathbf{1} - W^\dagger W} \end{pmatrix} \tag{107}$$

where now

$$W = \chi(\eta)B \tag{108}$$

and $\chi(\eta) = \sin(2|\eta|)/(\sqrt{2}|\eta|)$.

The coherent state $|\eta\rangle$ is then simply

$$|\eta\rangle = M(\eta)|0\rangle = (\cos 2|\eta| \quad -\eta_2^*\chi(\eta) \quad -\eta_1^*\chi(\eta) \quad \eta_1\chi(\eta) \quad -\eta_2\chi(\eta))^T. \tag{109}$$

Relation (101) provides the projective coordinates, which are given by

$$w_i = \frac{\eta_i \sin|\eta|}{|\eta|} \qquad i = 1, 2. \tag{110}$$

In these coordinates, the coherent state, now denoted $|w\rangle$, is written

$$|w\rangle = (1 - 2|w|^2 \quad -w_2^* k(w) \quad -w_1^* k(w) \quad w_1 k(w) \quad -w_2 k(w))^T \tag{111}$$

where $|w|^2 = |w_1|^2 + |w_1|^2$ and $k(w) = \sqrt{2 - 2|w|^2}$. It is easy to see that these states are normalized, $\langle w|w\rangle = 1$. It is also easy to see (using again the matrix representations) that indeed $\langle w|A|w\rangle = 0$ for any element A of the Lie algebra.

The vanishing of the average values of the generators does not imply a trivial dynamics, since a general element in the enveloping algebra will not vanish. The Casimir operators of the $su(2)$ subalgebras, for example, are

$$J_1 = H_1^2 + \left[E_4^+, E_4^-\right]_+ \qquad J_2 = H_2^2 + \left[E_2^+, E_2^-\right]_+ \tag{112}$$

and their average values are simply given by

$$\mathcal{J}_1 = \mathcal{J}_2 = |w|^2(1 - |w|^2) \tag{113}$$

where $\mathcal{J}_i = \langle w|J_i|w\rangle$. The agreement between the values of $\mathcal{J}_1$ and $\mathcal{J}_2$ is particular of this representation, and will not happen in general.

7. Conclusions

We have explicitly obtained coherent states for the unitary symplectic group $USp(4)$, in an arbitrary irreducible representation. We have derived their normalization and the correspondent symplectic structure, which together with the expectation values of the algebra elements provides a classical limit for quantum systems with symplectic symmetry. In the case of symmetric representations we obtained an explicit resolution of unity and also canonical coordinates that bring the Poisson bracket to its standard form.

We have not tackled the problem of the diagonal representation of operators (sometimes called upper symbols or P-functions). Such representation is less trivial than the one using expectation values (sometimes called lower symbols or Q-functions), because upper symbols may fail to exist and (when they exist) are not unique in general. Interesting observations in this respect appear in [1] and a recent detailed analysis can be found in [29]. We just want to mention that the coherent states associated with the quotient $USp(4)/[SU(2) \times SU(2)]$ were seen to provide only the trivial representation of the group, because all the lower symbols vanished. How this is related to the existence and unicity of the diagonal representation must be investigated (we note that the states defined in section 6.1 belong to case A defined in [29], while the ones defined in (38) belong to their case B).

An interesting application of coherent states is the investigation of the classical limit of non-integrable quantum systems [23]. A Hamiltonian consisting of an integrable term plus a non-integrable perturbation, $H = H_0 + \epsilon V$, may display different types of dynamics depending on the value of ϵ. Using the formalism described here one could, in principle, follow the transition from integrability to chaos in both the quantum and the classical levels. We will explore these matters in a subsequent work.

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